

# Relativity group for noninertial frames in Hamilton's mechanics

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## Abstract

The group  $\mathcal{E}(3) = \mathcal{SO}(3) \otimes_s \mathcal{T}(3)$ , that is the homogeneous subgroup of the Galilei group parameterized by rotation angles and velocities, defines the continuous group of transformations between the frames of inertial particles in Newtonian mechanics. We show in this paper that the continuous group of transformations between the frames of noninertial particles following trajectories that satisfy Hamilton's equations is given by the Hamilton group  $\mathcal{Ha}(3) = \mathcal{SO}(3) \otimes_s \mathcal{H}(3)$  where  $\mathcal{H}(3)$  is the Weyl-Heisenberg group that is parameterized by rates of change of position, momentum and energy, i.e. velocity, force and power. The group  $\mathcal{E}(3)$  is the inertial special case of the Hamilton group.

## 1 Introduction

It is very well known that Galilean relativity of inertial frames with relative velocity and rotation is described by the Euclidean group that is the homogeneous subgroup of the full Galilei group<sup>1</sup>. We review, in the following section, the derivation of the action of the Euclidean group on frames of a particle in Newtonian space-time from the assumption of invariance of a Newtonian time line element and invariance of length in the inertial rest frame. The group multiplication law gives the usual Newtonian addition of velocity. The diffeomorphisms of the space-time with these invariants are the straight lines trajectories of an inertial particle.

The group of transformations between frames of particles following noninertial trajectories also has an invariant Newtonian time line element and invariance of length in the inertial rest frame. We use the Hamilton formulation on extended phase space with position, time, momentum and energy degrees of freedom and therefore must also have invariance of the symplectic metric. Using the same method as reviewed for the Euclidean group, this results in the Hamilton group that is parameterized by rotation angles, and rates of change of

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<sup>1</sup>The full Galelei group also includes translations in position and time

position, momentum and energy with time, i.e. velocity, force and power. The group multiplication law results in the usual Newtonian addition of velocities and force. The diffeomorphisms with these invariants must satisfy Hamilton's equations of motion. The power transformation law has terms that integrate to those terms in the Hamiltonian that are required in noninertial frames. The homogeneous subgroup of the Galilei group is the inertial special case of the Hamilton group.

## 2 Newtonian inertial frames

The Newtonian space-time  $\mathbb{M} \simeq \mathbb{R}^{n+1}$  has coordinates  $x = (q, t)$  where  $q \in \mathbb{R}^n$  are the  $n$  position co-ordinates and  $t \in \mathbb{R}$  is the time coordinate. The usual physical case corresponds to  $n = 3$ . A frame in the cotangent space at a point  $x$ ,  $T_x^* \mathbb{M}$ , has a basis  $dx = (dq, dt)$ . The action of the general linear group element  $\Gamma \in \mathcal{GL}(n+1, \mathbb{R})$  on the cotangent space, suppressing the indices and using basic matrix notation is

$$d\tilde{x} = \Gamma \cdot dx, \quad (1)$$

where  $\Gamma$  is a nonsingular  $(n+1) \times (n+1)$  real matrix and  $dx$  is a column vector.

The line element may be written as

$$ds^2 = dt^2 = \eta^\circ_{ab} dx^a dx^b = {}^t dx \cdot \eta^\circ \cdot dx, \quad (2)$$

where the indices  $a, b.. = 0, 1..n$  and  $\eta$  is an  $(n+1) \times (n+1)$  matrix

$$\eta^\circ = \begin{pmatrix} 0_{n \times n} & 0_{1 \times n} \\ 0_{n \times 1} & 1 \end{pmatrix}. \quad (3)$$

In this expression,  $0_{n \times m}$  is an  $n \times m$  zero matrix. The condition that the line element is invariant under the action of the group is

$$dt^2 = {}^t dx \cdot \eta^\circ \cdot dx = d\tilde{t}^2 = {}^t (\Gamma \cdot dx) \cdot \eta^\circ \cdot \Gamma \cdot dx, \quad (4)$$

and therefore

$$\eta^\circ = {}^t \Gamma \cdot \eta^\circ \cdot \Gamma. \quad (5)$$

We may write  $\Gamma$  as an  $(n+1) \times (n+1)$  matrix of the form

$$\Gamma = \begin{pmatrix} R & v \\ w & \epsilon \end{pmatrix} \quad (6)$$

with  $R$  an  $n \times n$  submatrix,  $\epsilon \in \mathbb{R}$  and  $v, w \in \mathbb{R}^n$  with  $v$  a column vector and  $w$  a row vector. Equation (5) results in the expression

$$\eta^\circ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^t R & {}^t w \\ {}^t v & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & v \\ w & \epsilon \end{pmatrix} = \begin{pmatrix} {}^t w w & {}^t w \epsilon \\ \epsilon w & \epsilon^2 \end{pmatrix}, \quad (7)$$

where the dimensions of the zero matrices are now implicit. It follows directly that  $w = 0$  and  $\epsilon = \pm 1$ .

The group multiplication and inverse property is realized by matrix multiplication and inverse and a direct calculation shows it defines the matrix group with group multiplication and inverse given by

$$\begin{aligned}\Gamma(\epsilon, R, v) &= \Gamma(\epsilon'', R'', v'') \cdot \Gamma(\epsilon', R', v') = \Gamma(\epsilon''\epsilon', R'' \cdot R', R'' \cdot v' + \epsilon'v''), \\ \Gamma^{-1}(\epsilon, R, v) &= \Gamma(\epsilon, R^{-1}, -\epsilon R^{-1} \cdot v).\end{aligned}\quad (8)$$

The requirement that  $\det \Gamma \neq 0$  requires that  $\det R \neq 0$  and therefore  $R \in \mathcal{GL}(n, \mathbb{R})$ . The group elements  $\Gamma(1, R, 0)$  define the natural embedding of  $\mathcal{GL}(n, \mathbb{R})$  into  $\mathcal{GL}(n+1, \mathbb{R})$ . The elements  $\Gamma(\epsilon, I_n, 0) \in \mathcal{D}_2$  where  $\mathcal{D}_2$  is the two element discrete group of time reversal. Note that  $\Gamma(\epsilon, R, 0) \in \mathcal{D}_2 \otimes \mathcal{GL}(n, \mathbb{R})$ .

The *translation* group is defined to be the matrix Lie group  $\mathcal{T}(n)$  that is isomorphic to  $\mathbb{R}^n$  considered to be an abelian group under addition,  $\mathcal{T}(n) \simeq (\mathbb{R}^n, +)$ . The group elements  $\Gamma(1, I_n, v)$ , with  $I_n$  the  $n \times n$  unit matrix, define elements of the translation group  $\mathcal{T}(n)$  with group composition

$$\begin{aligned}\Gamma(1, I_n, v) &= \Gamma(1, I_n, v'') \cdot \Gamma(1, I_n, v') = \Gamma(1, I_n, v' + v''), \\ \Gamma^{-1}(1, I_n, v) &= \Gamma(1, I_n, -v).\end{aligned}\quad (9)$$

In this case, the translation group is parameterized by velocity: the translation are in velocity space rather than position space. The automorphisms of this translation subgroup are

$$\Gamma(\epsilon', R', v') \cdot \Gamma(1, I_n, v) \cdot \Gamma^{-1}(\epsilon', R', v') = \Gamma(1, I_n, \epsilon' R' \cdot v), \quad (10)$$

and therefore the translation group is a normal subgroup. The intersection of this translation subgroup with the subgroup  $\mathcal{D}_2 \otimes \mathcal{GL}(n, \mathbb{R})$  is the identity and the union is the entire group. Therefore, the group is the extended inhomogeneous general linear group

$$\mathcal{IGL}(n, \mathbb{R}) \simeq \mathcal{D}_2 \otimes_s \mathcal{IGL}(n, \mathbb{R}) \simeq \mathcal{D}_2 \otimes_s \mathcal{GL}(n, \mathbb{R}) \otimes_s \mathcal{T}(n). \quad (11)$$

The above group does not leave length,  $dq^2$ , invariant in the rest frame. The rest frame is the special case where  $v = 0$ . Requiring that the line element  $dq^2$  is invariant in the inertial rest frame

$$dq^2 = {}^t dx \cdot \eta^q \cdot dx = {}^t dx \cdot {}^t \Gamma(\epsilon, R, 0) \cdot \eta^q \cdot \Gamma(\epsilon, R, 0) \cdot dx, \quad (12)$$

results in the condition  $\eta^q = {}^t \Gamma(\epsilon, R, 0) \cdot \eta^q \cdot \Gamma(\epsilon, R, 0)$ . This may be written in matrix notation as

$$\eta^q = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^t R & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & \epsilon \end{pmatrix} = \begin{pmatrix} {}^t R \cdot R & 0 \\ 0 & 0 \end{pmatrix}, \quad (13)$$

where  $I_n$  is the  $n \times n$  unit matrix. This requires that  ${}^t R = R^{-1}$  and therefore  $R \in \mathcal{O}(n)$ .

Matrices of the form  $\Gamma(\epsilon, R, v)$  with  $\epsilon = \pm 1$ ,  $v \in \mathbb{R}^n$  and  $R \in \mathcal{O}(n)$  are elements of the matrix Lie group, generally called the extended Euclidean group,  $\hat{\mathcal{E}}(n)$  where

$$\hat{\mathcal{E}}(n) \simeq \mathcal{D}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{T}(n). \quad (14)$$

The group multiplication and inverse is given by (8) with  $R \in \mathcal{O}(n)$ .

The orthogonal group may be written as the semidirect product of the special orthogonal group and a 2 element discrete parity group  $\tilde{\mathcal{D}}_2$  as  $\mathcal{O}(n) = \tilde{\mathcal{D}}_2 \otimes_s \mathcal{SO}(n)$ . Define  $\varsigma \in \mathcal{D}_4 = \mathcal{D}_2 \otimes \tilde{\mathcal{D}}_2$  as the 4 element parity, time reversal group with elements

$$\varsigma = \begin{pmatrix} \tilde{\epsilon} I_n & 0 \\ 0 & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1, \tilde{\epsilon} = \pm 1. \quad (15)$$

The extended Euclidean group (14) may be written as

$$\hat{\mathcal{E}}(n) \simeq \mathcal{D}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{T}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{SO}(n) \otimes_s \mathcal{T}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{E}(n). \quad (16)$$

where  $\mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{T}(n)$  is the Euclidean matrix Lie group that is the homogeneous subgroup of the Galilei group.

Consider a transformation  $\tilde{x} = \varphi(x)$  that preserves the Newtonian time line element and length in the rest frame. The matrix of the Jacobian of the transformation is therefore an element of the group,  $[\frac{\partial \varphi(x)}{\partial x}]|_x \in \mathcal{E}(n)$ . The discrete transformations do not need to be considered as the  $\varphi$  are continuous and therefore only the continuous group is required. Furthermore, we can rotate the coordinates so that the rotation group need not be considered. Then, the Jacobian is an element of the translation normal subgroup of the Euclidean group,

$$\left( \frac{\partial \varphi^a(x)}{\partial x^a} \right) = \begin{pmatrix} \frac{\partial \varphi^i(t, q)}{\partial q^j} & \frac{\partial \varphi^i(t, q)}{\partial t} \\ \frac{\partial \varphi^0(t, q)}{\partial q^j} & \frac{\partial \varphi^0(t, q)}{\partial t} \end{pmatrix} = \begin{pmatrix} \delta_{i,j} & v^i \\ 0 & 1 \end{pmatrix}, \quad (17)$$

where in this expression indices  $i, j = 1, \dots, n$  are explicit. With  $v$  constant and ignoring trivial integration constants, the transformation equations may be integrated to

$$\tilde{q}^i = \varphi^i(q, t) = q^i + v^i t, \quad \tilde{t} = \varphi^0(q, t) = t. \quad (18)$$

The Euclidean group defines the transformation between inertial frames in classical Newtonian mechanics. The Euclidean group leaves invariant  $dt$  and therefore in Newtonian physics there is the notion of absolute time that all observers agree on. As the frame is inertial, the rate of change of momentum is zero and the motion is *uniform*. Correspondingly, velocity is simply additive as given by the group laws (8,9).

### 3 Hamilton group: Newtonian noninertial frames

The method used above to derive the Euclidean group for inertial frames may be applied directly to obtain the group of transformations between general non-inertial frames in Hamilton's mechanics. Again, we require invariance of the

Newtonian time line element and also that length is invariant in the inertial rest frame. As we are using the Hamilton formulation, we also require invariance of the symplectic metric.

For the noninertial case with non zero rate of change of momentum and position between frames of particle states, consider the space  $\mathbb{P} \simeq \mathbb{R}^{2n+2}$  that has coordinates  $z = (p, q, e, t)$ .  $p, q \in \mathbb{R}^n$  are the  $n$  momentum and  $n$  position co-ordinates,  $e \in \mathbb{R}$  is the energy coordinate and  $t \in \mathbb{R}$  the time coordinate.  $n = 3$  is the physical case. A frame at a point in the cotangent space  $T_z^*\mathbb{P}$  has a basis  $dz = (dp, dq, de, dt)$ . The action of an element of  $\Phi \in \mathcal{GL}(2n+2, \mathbb{R})$  on the frame is

$$d\tilde{z} = \Phi \cdot dz, \quad (19)$$

where  $\Phi$  is a nonsingular  $2(n+1) \times 2(n+1)$  matrix.

As in the Euclidean case, we consider the subgroup that leaves invariant the Newtonian time line element  $ds^2 = dt^2$ .

$$d\tilde{z} = dt^2 = {}^t dz \cdot \eta^\circ \cdot dz, \quad (20)$$

where  $\eta^\circ$  is now a  $2(n+1) \times 2(n+1)$  singular matrix. In addition, we again require that the length  $dq^2$  be invariant in the inertial rest frame of the particle.

Hamilton's mechanics has an additional invariant, the symplectic metric  $-de \wedge dt + \delta_{i,j} dp^i \wedge dq^j$  where  $i, j.. = 1, ..n$ .

Let  $\Phi$  be an  $(2n+2) \times (2n+2)$  nonsingular real matrix written in terms of submatrices as

$$\Phi = \begin{pmatrix} A & b & w \\ {}^t c & a & r \\ {}^t d & f & \epsilon \end{pmatrix}, \quad (21)$$

where  $A$  is a  $2n \times 2n$  real matrix,  $w, b, c, d \in \mathbb{R}^n$  and  $a, r, f, \epsilon \in \mathbb{R}$ . From the analysis in the previous section, we have immediately that the invariance of the Newtonian time line element requires  $\epsilon = \pm 1$ ,  $d, f = 0$  so that  $\Phi \in \mathcal{IGL}(2n+1, \mathbb{R})$ ,

$$\Phi = \begin{pmatrix} A & b & w \\ {}^t c & a & r \\ 0 & 0 & \epsilon \end{pmatrix}, \epsilon = \pm 1. \quad (22)$$

Furthermore we know that the requirement that the symplectic metric is invariant requires that  $\Phi \in \mathcal{Sp}(2n+2)$ . The group  $\mathcal{G}$  that leaves both the Newtonian line element and the symplectic group invariant is the intersection of these two groups

$$\mathcal{G} \simeq \mathcal{Sp}(2n+2) \cap \mathcal{IGL}(2n+1) \quad (23)$$

This group may be explicitly calculated simply by applying the symplectic condition to the explicit form of the group elements  $\Phi \in \mathcal{IGL}(2n+1, \mathbb{R})$  given in (22).

In the basis  $\{dp, dq, de, dt\}$  the matrix for symplectic metric  ${}^t dz \cdot \zeta \cdot dz$  has the form

$$\zeta = \begin{pmatrix} \zeta^0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (24)$$

where  $\zeta^\circ$  is the  $2n \times 2n$  matrix

$$\zeta^\circ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad (25)$$

with  $I_n$  the  $n \times n$  identity matrix.

Next, impose the condition that symplectic metric is invariant  ${}^t\Phi \cdot \zeta \cdot \Phi = \zeta$  using  $\Phi$  defined in (22)

$$\begin{pmatrix} {}^tA \cdot \zeta^\circ \cdot A & {}^tA \cdot \zeta^\circ \cdot b & {}^tA \cdot \zeta^\circ \cdot w - \epsilon c \\ {}^tb \cdot \zeta^\circ \cdot A & {}^tb \cdot \zeta^\circ \cdot b & {}^tb \cdot \zeta^\circ \cdot w - \epsilon a \\ {}^tw \cdot \zeta^\circ \cdot A + \epsilon \quad {}^tc & {}^tw \cdot \zeta^\circ \cdot b + \epsilon a & {}^tw \cdot \zeta^\circ \cdot w \end{pmatrix} = \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (26)$$

First,  ${}^tA \cdot \zeta^\circ \cdot A = \zeta^\circ$  implies that  $A \in \mathcal{Sp}(2n)$ . It follows from  ${}^tb \cdot \zeta^\circ \cdot A = 0$  and  ${}^tA \cdot \zeta^\circ \cdot b = 0$  that  $b = 0$ . Note that  ${}^tw \cdot \zeta^\circ \cdot w \equiv 0$  as  $\zeta^\circ$  is antisymmetric. Then from the terms  ${}^tw \cdot \zeta^\circ \cdot b + \epsilon a = 1$  and  ${}^tb \cdot \zeta^\circ \cdot w - \epsilon a = -1$ , with  $b = 0$  we have  $a = \epsilon$ . Finally, the remaining equations are

$$c = \epsilon \quad {}^tA \cdot \zeta^\circ \cdot w, \quad {}^tc = -\epsilon \quad {}^tw \cdot \zeta^\circ \cdot A \quad (27)$$

Noting that  ${}^t\zeta^\circ = -\zeta^\circ$ , these two equations are equivalent and therefore the matrix  $\Phi$  takes the form

$$\Phi(\epsilon, A, w, r) = \begin{pmatrix} A & 0 & w \\ -\epsilon & {}^tw \cdot \zeta^\circ \cdot A & \epsilon & r \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \quad (28)$$

It follows straightforwardly that this is a matrix group with the group multiplication realized by matrix multiplication and the group inverse by matrix inverse

$$\begin{aligned} \Phi(\epsilon, A, w, r) &= \Phi(\epsilon'', A'', w'', r'') \cdot \Phi(\epsilon', A', w', r'), \\ \Phi^{-1}(\epsilon, A, w, r) &= \Phi(\epsilon, A^{-1}, -\epsilon A^{-1} \cdot w, -r). \end{aligned} \quad (29)$$

where

$$\begin{aligned} \epsilon &= \epsilon'' \epsilon', \quad A = A'' \cdot A', \\ w &= \epsilon' w'' + A'' \cdot w', \\ r &= \epsilon'' r' + \epsilon' r'' - \epsilon'' {}^tw'' \cdot \zeta^\circ \cdot A'' \cdot w'. \end{aligned} \quad (30)$$

It is clear that  $\Phi(1, A, 0, 0) \in \mathcal{Sp}(2n)$  and  $\Phi(\epsilon, I_{2n}, 0, 0) \in \mathcal{D}_2$  and further that  $\Phi(\epsilon, A, 0, 0) \in \mathcal{D}_2 \otimes \mathcal{Sp}(2n)$

$$\Phi(1, A, 0, 0) = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Phi(\epsilon, I_{2n}, 0, 0) = \begin{pmatrix} I_{2n} & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \quad (31)$$

The elements of the discrete group  $\mathcal{D}_2$  change the sign of the time and energy degrees of freedom together and the elements of the symplectic group  $\mathcal{Sp}(2n)$  are the usual symplectic transformations on the position and momentum degrees of freedom.

Note also that for  $A'' = A' = I_{2n}$  and  $\epsilon'' = \epsilon' = 1$  that the group multiplication law reduces to

$$\begin{aligned}\Phi(1, I_{2n}, w, r) &= \Phi(1, I_{2n}, w'', r'') \cdot \Phi(1, I_{2n}, w', r') \\ \Phi^{-1}(1, I_{2n}, w, r) &= \Phi(1, I_{2n}, -w, -r)\end{aligned}\quad (32)$$

where

$$\begin{aligned}w &= w'' + w', \\ r &= r' + r'' - {}^t w'' \cdot \zeta^\circ \cdot w'.\end{aligned}\quad (33)$$

and therefore  $\Upsilon(w, r) = \Phi(1, I_{2n}, w, r)$  defines a subgroup  $\mathcal{H}(n)$  that has the group multiplication and inverse given by (32) with  $w \in \mathbb{R}^{2n}$  and  $r \in \mathbb{R}$ . This group is the Weyl-Heisenberg group.

The automorphisms of this subgroup are

$$\begin{aligned}\Phi(\epsilon, A, w, r) &= \Phi(\epsilon', A', w', r') \cdot \Phi(1, I_{2n}, w'', r'') \cdot \Phi^{-1}(\epsilon', A', w', r') \\ &= \Phi(1, I_{2n}, \epsilon' A' \cdot w'', r'' - {}^t w' \cdot \zeta^\circ \cdot w'' + {}^t w'' \cdot \zeta^\circ \cdot w')\end{aligned}\quad (34)$$

Therefore  $\mathcal{H}(n)$  is a normal subgroup. The union of  $\mathcal{H}(n)$  with  $\mathcal{D}_2 \otimes Sp(2n)$  is the full group and the intersection is the identity. Thus we have the result that the group  $\mathcal{G}$  that leaves the symplectic metric and the Newtonian time line element invariant is.

$$\mathcal{G} \simeq Sp(2n+2) \cap \mathcal{IGL}(2n+1) \simeq \mathcal{H}\hat{S}p(2n) = \mathcal{D}_2 \otimes_s Sp(2n) \otimes_s \mathcal{H}(n) \quad (35)$$

It is shown in [1] that this group is the group of linear automorphisms of the Weyl-Heisenberg group and therefore is the maximal group with a Weyl-Heisenberg normal subgroup .

### 3.1 Weyl-Heisenberg group

The notational change  $w = (f, v)$  with  $f, v \in \mathbb{R}^n$  enables the group operations of the Weyl-Heisenberg group to be written in the form

$$\begin{aligned}\Upsilon(f, v, r) &= \Upsilon(f'', v'', r'') \cdot \Upsilon(f', v', r') \\ &= \Upsilon(f'' + f', v'' + v', r'' + r' - f'' \cdot v' + v'' \cdot f'), \\ \Upsilon^{-1}(f, v, r) &= \Upsilon(-f, -v - r).\end{aligned}\quad (36)$$

$\Upsilon(f, v, r)$  may be realized by the matrix group [2]

$$\Upsilon(f, v, r) = \begin{pmatrix} I_n & 0 & 0 & f \\ 0 & I_n & 1 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

and it can be directly verified that matrix multiplication and inverse realizes the group operations given in (36).

The Weyl-Heisenberg group itself is the semidirect product of two translation groups

$$\mathcal{H}(n) \simeq \mathcal{T}(n) \otimes_s \mathcal{T}(n+1). \quad (38)$$

This may be shown as follows. Consider the group multiplication in (36) with  $f = r = 0$ ,

$$\begin{aligned}\Upsilon(0, v'', 0) \cdot \Upsilon(0, v', 0) &= \Upsilon(f, v, r) = \Upsilon(0, v'' + v', 0), \\ \Upsilon^{-1}(0, v, 0) &= \Upsilon(0, -v, 0).\end{aligned}\quad (39)$$

Thus,  $\Upsilon(0, v, 0) \in \mathcal{T}(n)$ . As in the Euclidean case, these translations are parameterized by velocity. Furthermore, with  $v = 0$  and  $f, r \neq 0$ , we have

$$\begin{aligned}\Upsilon(f'', 0, r'') \cdot \Upsilon(f', 0, r') &= \Upsilon(f, v, r) = \Upsilon(f'' + f', 0, r'' + r'), \\ \Upsilon^{-1}(f, 0, r) &= \Upsilon(-f, 0, -r).\end{aligned}\quad (40)$$

and therefore  $\Upsilon(f, 0, r) \in \mathcal{T}(n+1)$ . These translations are parameterized by force and power.

Finally, a special case of the automorphism given in (34) gives

$$\Upsilon(f', v', r') \cdot \Upsilon(f, 0, r) \cdot \Upsilon^{-1}(f', v', r') = \Upsilon(f, 0, r - 2f \cdot v'). \quad (41)$$

The translation subgroup  $\Upsilon(f, 0, r) \in \mathcal{T}(n+1)$  of  $\mathcal{H}(n)$  is therefore a normal subgroup. It may be shown that this is not the case for the translation subgroup  $\Upsilon(0, v, 0) \in \mathcal{T}(n)$ . Therefore, the Weyl-Heisenberg group is the semidirect product given in (38).

The Weyl-Heisenberg group that appears as a subgroup of the group of transformations between noninertial frames is parameterized by velocity, force and power. From the group multiplication given in (36), velocity and force are simply additive as expected in Newtonian mechanics. This identification will become clearer in the following section as well as the meaning of the power transformation law.

### 3.2 Hamilton's equations

We consider now the transformations  $\tilde{z} = \varphi(z) = \varphi(p, q, e, t)$  that leave the symplectic metric and the Newtonian line element  $dt^2$  invariant. From (35), the continuous group leaving this invariant is  $\mathcal{HSp}(2n)$ . The Jacobian of the transformation,  $\frac{\partial \varphi(z)}{\partial z}$ , must be an element of this group. Consider first the case where the Jacobian is an element of the  $\mathcal{H}(n)$  subgroup of  $\mathcal{HSp}(2n)$ .

$$\left[ \frac{\partial \varphi^\alpha(z)}{\partial z^\beta} \right] \Big|_z = \Upsilon(f, v, r) \quad (42)$$

Set  $z^\alpha = \{p^i, q^j, e, t\}$  with  $\alpha = 1, \dots, 2n+2$ ,  $i, j, \dots = 1, \dots, n$ . Then the above expression can be expanded out to

$$\begin{pmatrix} \frac{\partial \varphi^i(z)}{\partial p^j} & \frac{\partial \varphi^i(z)}{\partial q^j} & \frac{\partial \varphi^i(z)}{\partial e} & \frac{\partial \varphi^i(z)}{\partial t} \\ \frac{\partial \varphi^{n+i}(z)}{\partial p^j} & \frac{\partial \varphi^{n+i}(z)}{\partial q^j} & \frac{\partial \varphi^{n+i}(z)}{\partial e} & \frac{\partial \varphi^{n+i}(z)}{\partial t} \\ \frac{\partial \varphi^{2n+1}(z)}{\partial p^j} & \frac{\partial \varphi^{2n+1}(z)}{\partial q^j} & \frac{\partial \varphi^{2n+1}(z)}{\partial e} & \frac{\partial \varphi^{2n+1}(z)}{\partial t} \\ \frac{\partial \varphi^{2n+2}(z)}{\partial p^j} & \frac{\partial \varphi^{2n+2}(z)}{\partial q^j} & \frac{\partial \varphi^{2n+2}(z)}{\partial e} & \frac{\partial \varphi^{2n+2}(z)}{\partial t} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 & 0 & f^i \\ 0 & \delta_j^i & 0 & v^i \\ v^j & -f^j & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (43)$$



The solution of these equations requires the  $\varphi^\alpha$  to have the form

$$\begin{aligned}\tilde{p}^i &= \varphi^i(p, q, e, t) = p^i + \varphi_p^i(t), \\ \tilde{q}^i &= \varphi^{n+i}(p, q, e, t) = q^i + \varphi_q^i(t), \\ \tilde{e} &= \varphi^{2n+1}(p, q, e, t) = e + H(p, q, t), \\ \tilde{t} &= \varphi^{2n+2}(p, q, e, t) = t.\end{aligned}\tag{44}$$

In addition these equations must satisfy

$$\begin{aligned}\frac{\partial \varphi^{n+i}(z)}{\partial t} &= v^i = \frac{\partial \varphi^{2n+1}(z)}{\partial p^i}, & \frac{\partial \varphi^{2n+1}(z)}{\partial t} &= r, \\ \frac{\partial \varphi^i(z)}{\partial t} &= f^i = -\frac{\partial \varphi^{2n+1}(z)}{\partial q^i},\end{aligned}$$

that on substituting in (44) is Hamilton's equations

$$\frac{d\varphi_q^i(t)}{dt} = v^i = \frac{\partial H(p, q, t)}{\partial p^i}, \quad \frac{d\varphi_p^i(t)}{dt} = f^i = -\frac{\partial H(p, q, t)}{\partial q^i}, \quad \frac{\partial H(p, q, t)}{\partial t} = r.\tag{45}$$

From this result, the identification of  $v$  with velocity,  $f$  with force and  $r$  with power is clear. The group operation describes the addition of these quantities for transformations between frames associated with particles following trajectories that satisfy Hamilton's equations that are generally noninertial. The terms in the power transformation in the group multiplication law (56,32) integrate to the terms in the Hamiltonian required for noninertial frames.

On the other hand, if the Jacobian  $\frac{\partial \varphi(z)}{\partial z}$  is an element of the subgroup  $Sp(2n) \subset \mathcal{H}Sp(2n)$ , then the transformations are the usual canonical transformations on momentum-position space. These transformations leave invariant the symplectic metric  $\delta_{i,j} dp^i \wedge dq^j$  and Hamilton's equations.

Thus, from the condition that the Newtonian time line element  $dt^2$  and the condition that the symplectic metric  $\zeta$  are invariant on a  $2n+2$  dimensional space, we have derived Hamilton's equations on  $2n$  dimensional phase space and the invariance under the canonical transformations with Jacobians elements of  $Sp(2n)$ . However, viewed on the  $2n+2$  dimensional space, the transformation group is  $Sp(2n) \otimes_s \mathcal{H}(n)$ . This group transforms between the frames associated with particles following trajectories defined by Hamilton's equations that are generally noninertial.

### 3.3 The Hamilton group

Finally, we may also consider the invariance of the length line element

$$dq^2 = \delta_{ij} dq^i dq^j = {}^t dz \cdot \eta^q \cdot dz,\tag{46}$$

in the inertial rest frame as in the Euclidean case. The inertial rest frame is defined by  $v = f = r = 0$  and therefore

$${}^t \Phi(1, A, 0, 0, 0) \cdot \eta^q \cdot \Phi(1, A, 0, 0, 0) = \eta^q.\tag{47}$$

The  $2n \times 2n$  matrix  $A \in \mathcal{Sp}(2n)$  may be decomposed into the four  $n \times n$  submatrices  $A_{\mu,\nu}$  with  $\mu, \nu = 1, 2$ . In the  $2n + 2$  dimensional space, the  $\eta^q$  and  $\Phi(1, A, 0, 0, 0)$  are given by

$$\eta^q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Phi(1, A, 0, 0, 0) = \begin{pmatrix} A_{1,1} & A_{1,2} & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (48)$$

Then the invariance of the length line element (0) in the inertial rest frame results in

$$\begin{pmatrix} {}^t A_{2,1} \cdot A_{2,1} & {}^t A_{2,2} \cdot A_{2,1} & 0 & 0 \\ {}^t A_{2,1} \cdot A_{2,2} & {}^t A_{2,2} \cdot A_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (49)$$

where the dimensions of the zero submatrices are clear from the context. From this it follows that  $A_{2,2} = R \in \mathcal{O}(n)$  and  $A_{2,1} = 0$ .

The matrices  $A$  are elements of  $\mathcal{Sp}(2n)$  and therefore  ${}^t A \cdot \zeta^\circ \cdot A = \zeta^\circ$ . From this it follows that  $A^{-1} = -\zeta^\circ \cdot {}^t A \cdot \zeta^\circ$ . Writing the  $2n \times 2n$  matrices  $A$  in terms of the four  $n \times n$  submatrices  $A_{\mu,\nu}$ , we have

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}^{-1} = - \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} {}^t A_{1,1} & {}^t A_{2,1} \\ {}^t A_{1,2} & {}^t A_{2,2} \end{pmatrix} \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (50)$$

For the case  $A_{2,1} = 0$ , and  $A_{2,2} = R$  the inverse may be computed and therefore

$$\begin{pmatrix} A_{1,1}^{-1} & -A_{1,1}^{-1} \cdot A_{1,2} \cdot R^{-1} \\ 0 & R^{-1} \end{pmatrix} = \begin{pmatrix} {}^t R & {}^t A_{1,2} \\ 0 & {}^t A_{1,1} \end{pmatrix}. \quad (51)$$

Thus  $A_{1,1} = {}^t R^{-1}$ . Now, as  $R \in \mathcal{O}(n)$ , we have that  $R^{-1} = {}^t R$  and so  $A_{1,1} = R$ . The remaining condition is that

$$-{}^t R \cdot A_{1,2} \cdot R^{-1} \equiv {}^t A_{1,2} \quad \text{or} \quad {}^t R \cdot A_{1,2} \equiv -{}^t ({}^t R \cdot A_{1,2}). \quad (52)$$

As this is true for all  $R \in \mathcal{O}(n)$ , we have  $A_{1,2} = 0$ . This means that  $A$  is realized by the  $2n \times 2n$  matrices of the form

$$A = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, R \in \mathcal{O}(n), \quad (53)$$

and therefore  $A \in \mathcal{O}(n)$ .

This gives the result that the extended Hamilton group  $\hat{\mathcal{H}}a(n)$  is

$$\hat{\mathcal{H}}a(n) \simeq \mathcal{D}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{H}(n). \quad (54)$$

An element of the Hamilton group may be written explicitly in the  $(2n+2) \times (2n+2)$  matrix realization

$$\Phi(\epsilon, R, v, f, r) = \begin{pmatrix} R & 0 & 0 & f \\ 0 & R & 0 & v \\ v & -f & \epsilon & r \\ 0 & 0 & 0 & \epsilon \end{pmatrix}. \quad (55)$$

Again, as it is a matrix group, the group multiplication and inverse is given by matrix multiplication and inverse. Alternatively, these are just the special case of (0,0) with  $A$  given by (0) and  $w = (f, v)$

$$\begin{aligned} \Phi(\epsilon, R, f, v, r) &= \Phi(\epsilon'', R'', f'', v'', r'') \cdot \Phi(\epsilon', R', f', v', r'), \\ \Phi(\epsilon, R, f, v, r)^{-1} &= \Phi(\epsilon, R^{-1}, -\epsilon R \cdot f, -\epsilon R \cdot v, -\epsilon r), \end{aligned} \quad (56)$$

where

$$\begin{aligned} \epsilon &= \epsilon' \epsilon'', & R &= R'' \cdot R', \\ f &= \epsilon' f'' + R'' \cdot f', \\ v &= \epsilon' v'' + R'' \cdot v', \\ r &= \epsilon' r'' + \epsilon''(r' - f'' \cdot R'' \cdot v' + v'' \cdot R'' \cdot f'). \end{aligned} \quad (57)$$

These are the transformation equations for velocity  $v$ , force  $f$  and power  $r$  under the extended Hamilton group.

Note that for the inertial case with  $f = r = 0$ , that these reduce to

$$\begin{aligned} \Phi(\epsilon, R, 0, v, 0) &= \Phi(\epsilon'', R'', 0, v'', 0) \cdot \Phi(\epsilon', R', 0, v', 0) \\ &= \Phi(\epsilon' \epsilon'', 0, \epsilon' v'' + R'' \cdot v', 0), \\ \Phi(\epsilon, R, 0, v, 0)^{-1} &= \Phi(\epsilon, R^{-1}, 0, -\epsilon R \cdot v, 0). \end{aligned} \quad (58)$$

With the identification  $\Gamma(\epsilon, R, v) \simeq \Phi(\epsilon, R, 0, v, 0)$ , these are the group multiplication and inverse laws for the extended Euclidean group given in (0). Furthermore, noting that for  $f = v = 0$  that the Weyl-Heisenberg subgroup reduces to the translation group (0), we have that

$$\hat{\mathcal{E}}(n) \subset \hat{\mathcal{H}}a(n). \quad (59)$$

Thus the inertial group, that is the homogenous subgroup of the Galilei group, is a special case of the general noninertial Hamilton group.

Now, as in the Euclidean case, the orthogonal group can be decomposed into the direct product of the two element discrete parity group and the special orthogonal group,  $\mathcal{O}(n) \simeq \tilde{\mathcal{D}}_2 \otimes_s \mathcal{SO}(n)$ . The discrete two element parity group changes the sign of the position and momentum degrees of freedom together. The final step is to use this decomposition and again define the 4 element discrete group with elements  $\varsigma \in \mathcal{D}_4 \simeq \tilde{\mathcal{D}}_2 \otimes \mathcal{D}_2$  and restrict  $R \in \mathcal{SO}(n)$ . The  $(2n+2) \times (2n+2)$  matrix realization of the elements  $\varsigma \in \mathcal{D}_4$  are

$$\varsigma = \begin{pmatrix} \tilde{\epsilon} I_n & 0 & 0 & 0 \\ 0 & \tilde{\epsilon} I_n & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1, \tilde{\epsilon} = \pm 1 \quad (60)$$

The Hamilton group may then be written

$$\hat{\mathcal{H}}a(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{SO}(n) \otimes_s \mathcal{H}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{H}a(n) \quad (61)$$

where  $\mathcal{H}a(n) = \mathcal{SO}(n) \otimes_s \mathcal{H}(n)$ .

## 4 Discussion

We began the discussion in this paper by considering the group leaving ariance of spacial length in the inertial rest frame resulted in the extended Euclidean group of transformations that is the homogeneous subgroup of the Galilei group. The diffeomorphisms with a Jacobian that is an element of this group at a given point in the space-time define the usual linear inertial transformations. The extended Euclidean group defines the transformations between inertial frames in the Newtonian formulation.

We considered next the group that leaves invariant the Newtonian line element on a time, position, momentum, energy space formulation that also has a symplectic metric invariant. If again we require the invariance of length in the inertial rest frame, the extended Hamilton group of transformations results. The diffeomorphisms with a Jacobian that is an element of this group at a given point in the space-time define Hamilton's equations. Particles in classical mechanics follow trajectories that are defined by solutions to Hamilton's equations. The frames associated with these trajectories are in general noninertial. The extended Hamilton group therefore defines the transformations between general noninertial frames in the Hamilton formulation. The extended Euclidean transformations are a special case of the extended Hamilton transformations corresponding to the inertial case where the rate of change of momentum and energy are zero.

The Hamilton group multiplication defines the usual addition of velocity and force. The noninertial transformations of the power result in terms involving velocity and force appearing in the power transformation that integrate to the terms required in the Hamiltonian in a noninertial frame.

There is nothing fundamentally physical that distinguishes a particle in an inertial frame as apposed to a noninertial frame. The usual choice of inertial frames is simply a mathematical expediency to simplify the analysis. Furthermore, as inertial frames are related by a group, one expects that noninertial frames in the neighborhood of the inertial frame to likewise be related by a group. This is the Hamilton group. In this classical case, the noninertial formulation does not result in new physical consequences.

We know that the Euclidean group is the limit of small velocities,  $v/c \rightarrow 0$ , of the Lorentz group of special relativity. The Lorentz group defines transformations between frames of inertial particles in special relativity. Clearly, by the above arguments, there must be a group of transformations for noninertial frames in the relativistic case.<sup>2</sup> This group must have the Lorentz group as the

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<sup>2</sup>This is often assumed to be general relativity. The equivalence principle results in particles

inertial special case and contract in a well defined physical limit, that includes small velocities relative to  $c$ , to the Hamilton group. A group that satisfies these properties and the new physical consequences is discussed in [3, 4].

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following geodesics and so all particles in a purely gravitational system are locally inertial in the curved manifold. Consider the case with other forces where gravity is negligible and the problem of relativistic noninertial frames remains.